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# ANALYSIS OF THE HYPOTHESES USED WHEN CONSTRUCTING THE THEORY OF BEAMS AND PLATES†

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The solution of the two-dimensional problem of the theory of elasticity for a strip and the three-dimensional one for a plate are formulated by simple iterations and using asymptotic estimates with respect to a small parameter. These problems are solved in the literature by reducing the two-dimensional and three-dimensional problems to one-dimensional and two-dimensional ones, respectively, using the semi-inverse Saint-Venant's method [1, 2]. It is assumed that the solution obtained by the semi-inverse method has an error of the order of the relative size of the small domain of the applied self-balanced load. The treatment of the hypotheses, introduced in the semi-inverse method, as a selection of the respective initial approximation of the method of simple iterations enables the solution process to be formalized and provides an estimate of the error. The classical theory of beams and plates is supplemented by a solution of the boundary-layer type. The procedure is illustrated by solving the problem of a strip with an applied concentrated load. An additional solution for a rectangular plate, together with the solution of a biharmonic equation, enables three boundary conditions to be satisfied on each free end surface. © 2003 Elsevier Ltd. All rights reserved.

## 1. THE PROBLEM OF THE BENDING OF A STRIP

We will first consider the static antisymmetric problem for a long rectangular strip. We will relate it to the system of rectangular coordinates $x^*, z^*$ in such a way that $-l \leqslant x^* \leqslant l$, $-h \leqslant z^* \leqslant h$, ($l$ is half the length of the strip and $h$ is half its height). We will introduce dimensionless coordinates $x = x^*/l$, $z = z^*/h$ and dimensionless displacements $u = u^*/h$, $w = w^*/h$ along the $x^*, z^*$ axes respectively, and dimensionless stresses $\sigma_x = \sigma_x^*/E$, $\sigma_z = \sigma_z^*/E$, $\tau = \tau^*/E$ (the dimensional displacements, stresses and loads are denoted by an asterisk).

We will write the equations of the plane problem of the theory of elasticity, describing the stress-strain state of a strip of unit width, in the form

$$\frac{\partial u}{\partial z} = -\varepsilon\frac{\partial w}{\partial x} + 2(1+\nu)\tau, \quad \frac{\partial \sigma_z}{\partial z} = -\varepsilon\frac{\partial \tau}{\partial x}, \quad \varepsilon_x = \varepsilon\frac{\partial u}{\partial x}$$
$$\sigma_x = \varepsilon_x + \nu\sigma_z, \quad \varepsilon_z = (1-\nu^2)\sigma_z - \nu\varepsilon_x, \quad \frac{\partial w}{\partial z} = \varepsilon_z, \quad \frac{\partial \tau}{\partial z} = -\varepsilon\frac{\partial \sigma_x}{\partial x} \tag{1.1}$$

Here $\varepsilon = h/l$ is a small parameter.

We will seek the solution of Eq. (1.1) in the following way. Putting

$$w = w_0(x), \quad \tau = \tau_0(x) \tag{1.2}$$

in the first and second equation we will calculate $u_0$ and $\sigma_{z0}$ as known quantities of the zeroth approximation. We will then calculate $\varepsilon_{x0}$ and evaluate $\sigma_{x0}$ and $\varepsilon_{z0}$ using the relations of elasticity. Substituting these into the last two equations of Eq. (1.1) we will obtain $w_1$ and $\tau_1$ in the first approximation. The process of computing the next approximation can be continued.

Equation (1.1) reduces to the system of two equations

$$\frac{\partial^2 \tau}{\partial z^2} = \varepsilon^3\frac{\partial^3 w}{\partial x^3} - (2+\nu)\varepsilon^2\frac{\partial^2 \tau}{\partial x^2}, \quad \frac{\partial^2 w}{\partial z^2} = \nu\varepsilon^2\frac{\partial^2 w}{\partial x^2} - (1+\nu)^2\varepsilon\frac{\partial \tau}{\partial x} \tag{1.3}$$

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which we will write in the vector form

$$\mathbf{b} = A\mathbf{b} + \mathbf{c}z + \mathbf{d}$$

$$A = \int_0^z\int_0^z \left\| \begin{matrix} \varepsilon^3 \dfrac{\partial^3}{\partial x^3} & -(2+\nu)\varepsilon^2 \dfrac{\partial^2}{\partial x^2} \\ \nu\varepsilon^2 \dfrac{\partial^2}{\partial x^2} & -(1+\nu)^2 \varepsilon \dfrac{\partial}{\partial x} \end{matrix} \right\| dzdz$$

$$b = \{\tau, w\}, \quad c = \{\partial\tau/\partial z, \partial w/\partial z\}_{z=0}, \quad d = \{\tau, w\}_{z=0}$$

after double integration with respect to $z$.

We will seek the solution of the vector equations by the method of simple iterations.

$$\mathbf{b}_{n+1} = A\mathbf{b}_n + \mathbf{c}z + \mathbf{d}$$

($n$ is the number of the iteration).

If the operator $A$ is the compressing operator, the sequence of approximations $\mathbf{b}n$ converges to the exact solution of the equation as $n \to \infty$, irrespective of the choice of the initial approximation $\mathbf{b}_0$. There are numerous examples of the use of the method of simple iterations [3]. It is obvious that the sequence of the calculation of iterations described, originating from Eq. (1.1) and the sequence originating from Eq. (1.3) with a subsequent calculation of all the unknown quantities again originating from Eq. (1.1), are identical.

For convenience we will divide the initial approximation (1.2) into two

$$w = w_0(x), \quad \tau = \tau_0 = 0 \tag{1.4}$$

$$w = w_0 = 0, \quad \tau = \tau_0(x) \tag{1.5}$$

We will call the process of computing the unknown variables starting from (1.4) the $w$-process, while the process starting from (1.5) will be called the $\tau$-process.

The calculation of the components of the stress–strain sate gives in the $w$-process

$$\begin{gathered} w = w_0(x), \quad \tau_0 = 0, \quad u_0 = -\varepsilon w_0' z, \quad \varepsilon_{x0} = -\varepsilon^2 w_0'' z, \quad \sigma_{z0} = 0 \\ \sigma_{x0} = -\varepsilon^2 w_0'' z, \quad \varepsilon_{z0} = \nu\varepsilon^2 w_0'' z, \quad w_1 = \nu\varepsilon^2 w_0'' z^2/2, \quad \tau_1 = \varepsilon^3 w_0''' z^2/2 \end{gathered} \tag{1.6}$$

and in the $\tau$-process

$$\begin{gathered} \tau = \tau_0(x), \quad w_0 = 0, \quad u_0 = 2(1+\nu)\tau_0 z, \quad \sigma_{z0} = -\varepsilon\tau_0' z \\ \varepsilon_{x0} = 2(1+\nu)\varepsilon\tau_0' z, \quad \sigma_{x0} = (2+\nu)\varepsilon\tau_0' z, \quad \varepsilon_{z0} = -(1+\nu)^2 \varepsilon\tau_0' z \\ w_1 = -(1+\nu)^2 \varepsilon\tau_0' z^2/2, \quad \tau_1 = -(2+\nu)\varepsilon^2 \tau_0'' z^2/2 \end{gathered} \tag{1.7}$$

Differentiation with respect to $x$ is denoted by a prime.

We will assume that $w_0$ is a slowly varying function and $\tau_0$ is a rapidly varying one. This means that the application of the operator $\partial/\partial x$ to the function $w_0$ in an asymptotic sense is equivalent to multiplying this function by $\varepsilon^0$ and the application of the operator to $\tau_0$ is equivalent to an increase by a factor of $\varepsilon^{-1}$. We will call the first function the function of zero variability and the second the function of unit variability.

Let the longitudinal edges of the strip be free from shear stresses. Then Eqs (1.6) and (1.7) imply that the sum of the shear stresses of the $w$-process and the $\tau$-process gives the equation

$$\varepsilon^3 w_0'''/2 - \varepsilon^2 \tau_0''/k^2 + \tau_0 = 0, \quad k^2 = 2/(2+\nu)$$

for $z = \pm 1$.

Considering the last equation as an equation for $\tau_0(x)$, we will find it slowly varying particular solution

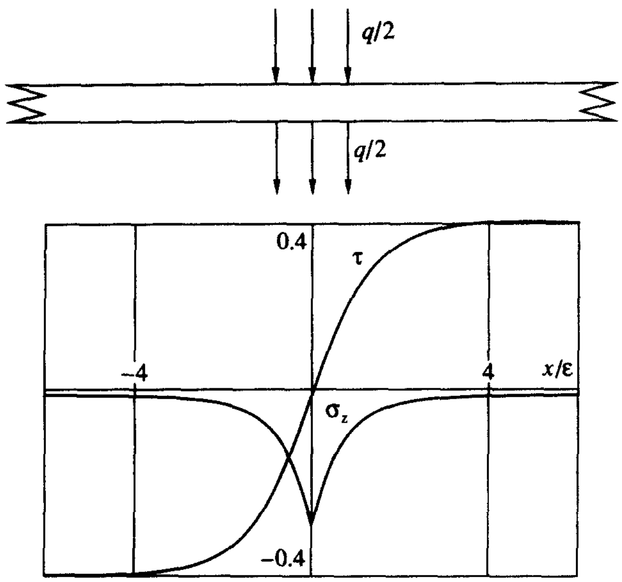


Fig. 1

$$\tau_0^p = -\varepsilon^3 w_0'''/2 \tag{1.8}$$

and its rapidly varying general solution

$$\tau_0^g = C_1 e^{-kx/\varepsilon} + C_2 e^{kx/\varepsilon} \tag{1.9}$$

The boundary conditions for $\sigma_z^*$, specified by the loading conditions, must now be satisfied on the longitudinal edges of the strip. It can be shown that if the strip is loaded by a distributed load $q^* = q^*(x^*)$, the boundary conditions are satisfied when there is no singular solution (1.9). The equation of classical theory

$$2/3\varepsilon^4 w_0^{IV} = q, \quad q = q^*/E$$

is obtained for $w_0$.

The displacement $w_0$ determined from this equation enables four conditions on the ends of the strip $x = \pm 1$ to be satisfied.

We will now consider the case of the loading of a freely supported strip with a load $q^*$ on part of a short length $2c$ (see the figure). We will replace the distributed load by a concentrated force $P^* = 2q^*c$. A solution of the type (1.9) arises close to the point of application of the force $P^*$ and decays rapidly with distance from the point $x = 0$. At the ends of the strip $x = \pm 1$ we have

$$\int_{-h}^{h} \tau^* \Big|_{x^* = \pm l} dz^* = \pm\frac{1}{2}P^*$$

These conditions, together with (1.6) and (1.7), lead to an equation for determining $w_0$

$$-\frac{2}{3}\varepsilon^3 w_0''' = \begin{cases} -P/2 & \text{when } -1 \le x \le 0 \\ P/2 & \text{when } 0 \le x \le 1 \end{cases} \tag{1.10}$$

Here $P = P^*/E$ is a dimensionless force.

Further, the conditions $\sigma_x = 0$ must be satisfied at the ends of the strip. The resultant expressions for the stresses on the left half of the strip (for $-1 \leqslant x \leqslant 0$) have the form

$$\begin{aligned} &\tau = -3/8P(1 - z^2)(1 - e^{kx/\varepsilon}), \quad \sigma_z = 1/8P(z^2 - 3)kze^{kx/\varepsilon} \\ &\sigma_x = 3/8P[-4\varepsilon^{-1}(1 + x) + (2 + \nu)e^{kx/\varepsilon}]z \end{aligned} \tag{1.11}$$

Similar expressions can be written for the right half of the strip.

The stress $\sigma_z$ has a small correction in the form of the second exponential term in square brackets, compared with the solution known from the strength of materials. Graphs of $\tau$ and $\sigma_z$ are shown in Fig. 1 for $P = 1$. The stress $\tau$ is given in the cross-section $z = 0$, and $\sigma_z$ is given in the cross-section $z = 1$.

## 2. THE CONVERGENCE OF THE METHOD

Solution (1.6), assuming that differentiation with respect to $x$ does not change the asymptotic order of the required quantities with respect to $\varepsilon$, gives

$$u \sim \varepsilon w, \quad (\varepsilon_x, \sigma_x, \varepsilon_z) \sim \varepsilon^2 w, \quad \tau \sim \varepsilon^3 w, \quad \sigma_z \sim \varepsilon^4 w \tag{2.1}$$

Solution (1.7), taking into account the fact that differentiation increases the required quantity by a factor of $\varepsilon$, gives zero orders for all the required quantities with respect to $\tau$. The quantities $w$ and $\tau$ are represented by series of the form

$$\begin{aligned} w &= w_0 + \nu\varepsilon^2 w_0'' z^2/2 + \ldots = w_0 + O(\varepsilon^2) \\ \tau &= \tau_0 - (2+\nu)\varepsilon^2 \tau_0'' z^2/2 + \ldots = \tau_0 + O(\varepsilon^0) \end{aligned} \tag{2.2}$$

The first series is asymptotic, and the second is not. Confining ourselves to the first term of the series for the calculation of $w$, we obtain an error of the order of $\varepsilon^2$ compared with unity. It is quite possible that, due to the non-asymptotic character of the series for $\tau$, several terms of the series should be taken into account when calculating the quantity $\tau$. In this connection we will slightly change the procedure for calculating the unknown quantities. We will rewrite Eq. (1.3) in the form

$$\frac{\partial^2 \tau}{\partial z^2} = \varepsilon^3 \frac{\partial^3 w}{\partial x^3} - (2+\nu)\frac{\partial^2 \tau}{\partial \xi^2}, \quad \frac{\partial^2 w}{\partial z^2} = \nu\varepsilon^2 \frac{\partial^2 w}{\partial x^2} - (1+\nu)\frac{\partial \tau}{\partial \xi} \tag{2.3}$$

introducing a new variable $\xi = x/\varepsilon$ so that the operation of differentiation of $\tau$ with respect to $\xi$ has asymptotic order $\varepsilon^0$. We will seek the unknown variables $w$ and $\tau$ in the form of expansions in asymptotic series in the small parameter $\varepsilon$, i.e.

$$w = \sum_{s=0} w_s \varepsilon^s, \quad \tau = \varepsilon^3 \sum_{s=0} \tau_s \varepsilon^s \tag{2.4}$$

where $w_s$, $\tau_s$ are quantities of order $\varepsilon^0$. Here we have taken into account that the relation

$$(\tau^p, \tau^g) \sim \varepsilon^3 w$$

follows from Eqs (1.8), (1.9) and (1.11).

We obtain equations for determining the coefficients of series (2.4) in ascending order of $s$

$$\frac{\partial^2 \tau_s}{\partial z^2} = \frac{\partial^3 w_s}{\partial x^3} - (2+\nu)\frac{\partial^2 \tau_s}{\partial \xi^2}, \quad \frac{\partial^2 w_s}{\partial z^2} = \frac{\partial^2 w_{s-2}}{\partial x^2} - (1+\nu)\frac{\partial \tau_{s-3}}{\partial \xi} \tag{2.5}$$

by substituting expansions (2.4) into Eq. (2.3) and equating coefficients of like powers of $\varepsilon$.

The quantities with negative indices should be considered to be equal to zero.

We will show that Eq. (2.5) is solvable in the zeroth approximation. Equations (2.5) have the form

$$\frac{\partial^2 \tau_0}{\partial z^2} - (2+\nu)\varepsilon^2 \frac{\partial^2 \tau_0}{\partial x^2} = \varepsilon^3 \frac{\partial^3 w_0}{\partial x^3}, \quad \frac{\partial^2 w_0}{\partial z^2} = 0 \tag{2.6}$$

for $s = 0$. Here the inverse replacement of $\xi$ by $x/\varepsilon$ has been made.

It follows from the second equation that $w_0$ is only a function of $x$. Taking this into consideration we will seek a particular solution of the first equation of (2.6) as a function of zero variability in $x$. In this case the second term on the left-hand side of the equation can be neglected as a small quantity of order $\varepsilon^2$. Then the particular solution $\tau_0^p$, taking into account the zero boundary conditions along the longitudinal edges of the strip, will be determined by the expression

$$\tau_0^p = w_0'''(z^2-1)/2 \tag{2.7}$$

The general solution which satisfies the condition that there are no shear stresses along the longitudinal edges has the form of a boundary layer

$$\tau_0^g = \sum_{n=1}^{\infty}\left[C_{1n}\exp\left(-\kappa_n\frac{x}{\varepsilon}\right)+C_{2n}\exp\left(\kappa_n\frac{x}{\varepsilon}\right)\right]\cos\frac{n\pi}{2}z, \quad \kappa_n = \frac{n\pi}{2(2+\nu)^{1/2}} \tag{2.8}$$

where $C_{1n}$, $C_{2n}$ are constants of integration.

We now have expressions for $\tau$ and $w$ obtained by two methods. One of these (2.2) is obtained by simple iterations originating from the initial approximations (1.4) and (1.5), while the second is obtained by asymptotic integration based on a knowledge of the asymptotic estimates (2.1), again obtained by simple iterations from the initial approximations (1.4) and (1.5). However, knowing the estimates, is was possible to depart from approximation (1.5) and obtain the solution by a different method without expanding $\tau$ in series in powers of $z$. Equation (2.4) is equivalent to choosing the initial approximation by the following method

$$w = w_0(x), \quad \tau = \tau_0(x,z) \tag{2.9}$$

i.e. it is assumed that only $w_0$ does not depend on the transverse coordinate.

In the theory of beams and plates the fact that $w$ is independent of the transverse coordinate is understood as the hypothesis of the non-deformability of the normal. The form (2.9) for $\tau$ indicates that, relative to $\tau$, no assumptions were made in the initial approximation, even though the relation of the orders of (2.1) between $\tau$ and $w$ was essentially used in the solution.

We will write the solution for the function $\tau$ considered in the previous part of the problem for a concentrated force

$$\tau = \varepsilon^3 w_0'''\frac{z^2-1}{2}+\sum_{n=1}^{\infty}C_{2n}\exp\left(\kappa_n\frac{x}{\varepsilon}\right)\cos\frac{n\pi z}{2}, \quad x<0$$

using relations (2.7), (2.8) and (2.1).

We will use the well-known expansion [4]

$$\varsigma^2-1 = -\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\frac{n\pi\varsigma}{2}, \quad -1\le\varsigma\le 1$$

(here and below $n$ only has odd values) and smooth the gap at the point $x = 0$. We write the solution in the following form

$$\tau = -\frac{12}{\pi^3}\left[1-\exp\left(\kappa_n\frac{x}{\varepsilon}\right)\right]\cos\frac{\pi z}{2}-\sum_{n=3}^{\infty}\frac{1}{n^3}\left[1-\exp\left(\kappa_n\frac{x}{\varepsilon}\right)\right]\cos\frac{n\pi z}{2} \tag{2.10}$$

The first term of the series is separated to compare it with expression (1.11), according to which, taking $\nu = 0.3$, we obtain

$$\tau = -0.375P(1-e^{0.94x/\varepsilon})(1-z^2)$$

We will leave only the first term on the right-hand side of Eq. (2.10), expand the cosine in a power series and retain the first two terms of the expansion. This gives the following approximate expression for $\tau$, obtained from the solution of the equation of the boundary layer

$$\tau = -0.387P(1-e^{1.04x/\varepsilon})(1-1.1z^2)$$

The last two expressions are fairly close and the estimate of the residual in Eq. (2.10) as a quantity of the order of 0.1 enables us to conclude that the initial approximation (1.5) produces a positive result.

We note that Eqs (2.6) are written for $s = 0$. We obtain the same equations for $s = 1$. Consequently, Eq. (2.6) has an accuracy of $\varepsilon^2$.

## 3. THE BENDING OF A PLATE

We will apply the above approach to the problem of the bending of a plate. We will superpose the median plane of a rectangular plate with the plane $x^*y^*$ of the Cartesian system of coordinates $x^*y^*z^*$. Let $a$ and $b$ be the dimensions of the plate along the $x^*$ and $y^*$ axes respectively, and let $2h$ be the thickness of the plate.

We will denote the dimensional displacements, stresses and load by an asterisk as in Section 1. We will choose $a$, $h$ and $E$ as the units of measurement of length, displacements and stresses, respectively. Then the region filled by the plate in dimensionless coordinates is given by the expressions

$$0 \le x \le 1, \quad 0 \le y \le b/a, \quad -1 \le z \le 1$$

We will write the equations describing the stress–strain state of the plate in the form

$$\begin{gathered}
\frac{\partial u}{\partial z} = -\varepsilon\frac{\partial w}{\partial x} + 2(1+\nu)\tau_{xz} \ (u \leftrightarrow v, x \leftrightarrow y), \quad \frac{\partial \sigma_z}{\partial z} = -\varepsilon\frac{\partial \tau_{xz}}{\partial x} - \varepsilon\frac{\partial \tau_{yz}}{\partial y} \\
\varepsilon_x = \varepsilon\frac{\partial u}{\partial x} \ (u \leftrightarrow v, x \leftrightarrow y), \quad 2(1+\nu)\tau_{xy} = \varepsilon\frac{\partial u}{\partial y} + \varepsilon\frac{\partial v}{\partial x} \\
\sigma_x = \frac{\varepsilon_x + \nu\varepsilon_y}{1-\nu^2} + \frac{\nu}{1-\nu}\sigma_z \ (x \leftrightarrow y) \\
\varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y) + \frac{(1+\nu)(1-2\nu)}{1-\nu}\sigma_z, \quad \frac{\partial w}{\partial z} = \varepsilon_z \\
\frac{\partial \tau_{xz}}{\partial z} = -\varepsilon\frac{\partial \sigma_x}{\partial x} - \varepsilon\frac{\partial \tau_{xz}}{\partial y} \ (x \leftrightarrow y)
\end{gathered} \tag{3.1}$$

The small parameter $\varepsilon = h/a$ is introduced here. The symbols $u \leftrightarrow v$, $x \leftrightarrow y$ or $x \leftrightarrow y$ after certain equations show that there should be two equations of a similar type; the second equation is obtained by circular replacement of the symbols shown.

Equation (3.1) is reduced to the following three equations by simple conversions

$$\begin{gathered}
\frac{\partial^2 \tau_{xz}}{\partial z^2} = \frac{1}{1-\nu^2}\varepsilon^3\frac{\partial}{\partial x}\Delta w + \varepsilon^2 L_{xy} \ (x \leftrightarrow y) \\
\frac{\partial^2 w}{\partial z^2} = \frac{\nu}{1-\nu}\varepsilon^2\Delta w - \frac{1+\nu}{1-\nu}\varepsilon\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right)
\end{gathered} \tag{3.2}$$

where

$$L_{xy} = -\frac{2-\nu}{1-\nu}\frac{\partial^2 \tau_{xz0}}{\partial x^2} - \frac{\partial^2 \tau_{xz0}}{\partial y^2} - \frac{1}{1-\nu}\frac{\partial^2 \tau_{yz0}}{\partial x \partial y} \ (x \leftrightarrow y)$$

the solution of which will be constructed by simple iterations, having chosen the quantities of the initial approximation as

$$w = w_0(x, y), \quad \tau_{xz} = \tau_{xz0}(x, y) \ (x \leftrightarrow y) \tag{3.3}$$

We will write the expressions for $\tau_{xz}$ and $\tau_{yz}$ in the first approximation

$$\tau_{xz} = \left(\frac{1}{1-\nu^2}\varepsilon^3\frac{\partial}{\partial x}\Delta w_0 + \varepsilon^2 L_{xy}\right)\frac{z^2}{2} + \tau_{xz0} \ (x \leftrightarrow y) \tag{3.4}$$

taking (3.3) into consideration and integrating the first two equations of (3.2) twice with respect to $z$.

Knowing $\tau_{xz}$ and $\tau_{yz}$ we will write the expression for $\sigma_z$, calculated using the third equation of system (3.1). We have

$$\sigma_z = \left[-\frac{1}{1-\nu^2}\varepsilon^4\Delta^2 w_0 - \varepsilon^3\frac{\partial L_{xy}}{\partial x} - \varepsilon^3\frac{\partial L_{yz}}{\partial y}\right]\frac{z^3}{6} - \varepsilon\left(\frac{\partial \tau_{xz0}}{\partial x} + \frac{\partial \tau_{yz0}}{\partial y}\right)z \tag{3.5}$$

The usual boundary conditions $\tau^*_{xz} = \tau^*_{yz} = 0$ for the problem of the bending of a plate by a transverse surface load $q^*$ for

$$z^* = \pm h,\ \sigma_z^* = \mp q^*/2 \text{ when } z^* = \pm h$$

must be satisfied on the faces of the plate.

We obtain the following equations for determining $w_0$ and $\tau_0$

$$\begin{gathered}\frac{1}{6}\left(-\frac{1}{1-\nu^2}\varepsilon^3\frac{\partial}{\partial x}\Delta w_0 + \varepsilon^2 L_{xy}\right) + \tau_{xz0} = 0 \quad (x \leftrightarrow y)\\ \frac{1}{6}\left(-\frac{1}{1-\nu^2}\varepsilon^4\Delta^2 w_0 - \varepsilon^3\frac{\partial L_{xy}}{\partial x} - \varepsilon^3\frac{\partial L_{yx}}{\partial y}\right) - \varepsilon\left(\frac{\partial \tau_{xz0}}{\partial x} + \frac{\partial \tau_{yz0}}{\partial y}\right) = -\frac{q}{2},\quad q = \frac{q^*}{E}\end{gathered} \tag{3.6}$$

on the faces of the plate corresponding to the boundary conditions by substituting the stresses (3.4) and (3.5) into the boundary conditions.

We multiply the first equation of (3.6) by 1/3 and differentiate it with respect to $x$, multiply the second equation by 1/3 and differentiate it with respect to $y$ and add it to the third equation. We obtain

$$\varepsilon\frac{\partial \tau_{xz0}}{\partial x} + \varepsilon\frac{\partial \tau_{yz0}}{\partial y} = \frac{3q}{4} \tag{3.7}$$

We will write the first two equations of system (3.6) as

$$\frac{1+\nu}{2(1-\nu)}\varepsilon^2\frac{\partial^2 \tau_{xz0}}{\partial x^2} + \varepsilon^2\frac{\partial^2 \tau_{xz0}}{\partial y^2} - \tau_{xz0} = \frac{1}{2(1-\nu^2)}\varepsilon^3\frac{\partial}{\partial x}\Delta w_0 - \frac{1}{4(1-\nu)}\varepsilon\frac{\partial q}{\partial x} \quad (x \leftrightarrow y) \tag{3.8}$$

taking Eq. (3.7) into consideration.

We note, considering this system as a system in the unknowns $\tau_{xz0}$ and $\tau_{yz0}$, that it splits into two independent equations.

Note that (3.5) reduces to the form

$$\sigma_z = \left[-\frac{1}{1-\nu^2}\varepsilon^4\Delta^2 w_0 + \frac{2-\nu}{1-\nu}\varepsilon^3\Delta\left(\frac{\partial \tau_{xz0}}{\partial x} + \frac{\partial \tau_{xz0}}{\partial y}\right)\right]\frac{z^3}{6} - \varepsilon\left(\frac{\partial \tau_{xz0}}{\partial x} + \frac{\partial \tau_{xz0}}{\partial y}\right)z$$

and this enables us, taking (3.7) into consideration, to write the third equation of system (3.6) as

$$-\frac{2}{3(1-\nu^2)}\varepsilon^4\Delta^2 w_0 = q - \frac{2-\nu}{2(1-\nu)}\varepsilon^2\Delta q \tag{3.9}$$

Equation (3.9) is the resolvent for the quantities of the $w$-process. All the remaining unknown variables can be expressed using (3.1) in terms of $w_0$. If the surface load $q$ has zero variability the second term on the right-hand side of the equation can be neglected and an equation for the deflection is derived which is identical with the equation of the classical theory of bending for plates. It has the form

$$D\Delta^{*2} = -q^*,\quad D = 2Eh^3/[3(1-\nu^2)]$$

after changing to dimensional variables, where $\Delta^*$ is the Laplace operator in the dimensional coordinates $x^*$ and $y^*$, and $D$ the flexural stiffness of the plate [5].

If $w_0$ and $q$ have zero variability, the estimate

$$w_0 \sim \varepsilon^{-4} q \tag{3.10}$$

is derived from Eq. (3.9).

Then the particular solutions of Eqs (3.8) are easily found

$$\tau^p_{xz0} = -\frac{1}{2(1-\nu^2)}\varepsilon^3 \frac{\partial}{\partial x}\Delta w_0 \quad (x \leftrightarrow y) \tag{3.11}$$

apart from quantities of the order of $\varepsilon^2$,

For the possibility of carrying out a further analysis we will assume that the components of the solution of Eq. (3.1) can be expanded in Fourier series in the coordinate $y$, while retaining the functions of zero variability. This enables us to neglect the second derivative with respect to $y$ as a quantity of the order of $\varepsilon^2$ in the first equation of Eq. (3.8). We will write the general solution of this equation as

$$\tau^g_{xz0} = C_0 e^{-kx/\varepsilon} + C_1 e^{-k(1-x)/\varepsilon}, \quad k^2 = 2(1-\nu)/(1+\nu) \tag{3.12}$$

where $C_0$, $C_1$ are constants of integration. The first term describes the solution which decays rapidly with distance from the edge $x = 0$ and the second term describes the solution which decays rapidly with distance from edge $x = 1$.

## 4. SATISFACTION OF THE CONDITIONS ON THE FRONT EDGES OF THE PLATE

We will consider the procedure for satisfying the boundary conditions using the solutions of the $w$ and $\tau$-processes with the example of the bending of a plate under a load $q = q_{mn}\sin(m\pi x)\sin(n\pi ay/b)$, where $m$ and $n$ are numbers of the order of $\varepsilon^0$. At the edges $y = 0$ and $y = a/b$ we will choose conditions $\sigma_y = w = u = 0$, corresponding to the conditions of free support in the theory of plates [5]. These conditions will be satisfied if we assume that $v$ and $\tau_{xy}$ are proportional to $\cos(m\pi ay/b)$, while the other unknown variables are proportional to $\sin(m\pi ay/b)$. We will consider the edges of the plate $x = 0$ and $x = 1$ to be free and require that the stresses on these should vanish

$$\sigma_x = \tau_{xy} = \tau_{xz} = 0 \quad \text{for} \quad x = 0.1 \tag{4.1}$$

These stresses in the plate, generated by the initial approximation (3.3), are easily calculated from Eqs (3.1) and have the form

$$\sigma_x(w_0) = -\frac{\varepsilon^2}{1-\nu^2}\left(\frac{\partial^2 w_0}{\partial x^2} + \nu\frac{\partial^2 w_0}{\partial y^2}\right)z, \quad \tau_{xy}(w_0) = -\frac{\varepsilon^2}{2(1+\nu)}\frac{\partial^2 w_0}{\partial x \partial y}z$$

$$\tau_{xz}(w_0) = \frac{\varepsilon^3}{1-\nu^2}\frac{\partial}{\partial x}\Delta w_0 \frac{z^2}{2}, \quad \tau_{xz} = \tau_{xz0}, \quad \tau_{xy}(\tau_{xz0}, \tau_{yz0}) = \varepsilon\frac{\partial \tau_{xz0}}{\partial y} + \varepsilon\frac{\partial \tau_{yz0}}{\partial x}$$

$$\sigma_x(\tau_{xz0}, \tau_{yz0}) = \frac{2-\nu}{1-\nu}\varepsilon\frac{\partial \tau_{xz0}}{\partial x} + \frac{\nu}{1-\nu}\varepsilon\frac{\partial \tau_{yz0}}{\partial y}$$

The symbols $w_0$, $\tau_{xz0}$, $\tau_{xz0}$ in brackets indicate quantities in terms of which the stresses are calculated.

We will write conditions (4.1), proceeding from these stresses and Eqs (3.10) and (3.11) symbolically, as

$$\begin{gathered}
\varepsilon^{-2}\sigma_x(w_0) + \varepsilon^{-1}\sigma_x(\tau^p_{xz0}, \tau^p_{yz0}) + \varepsilon^{\alpha}\sigma_x(\tau^g_{xz0}) = 0 \\
\varepsilon^{-2}\tau_{xy}(w_0) + \varepsilon^{-1}\tau_{xy}(\tau^p_{xz0}, \tau^p_{yz0}) + \varepsilon^{\alpha}\tau_{xy}(\tau^g_{xz0}) = 0 \\
\varepsilon^{-2}\tau_{xz}(w_0) + \varepsilon^{-2}\tau^p_{xz0} + \varepsilon^{\alpha-1}\tau^g_{xz0} = 0
\end{gathered} \tag{4.2}$$

The coefficients of different powers of $\varepsilon$ are of the order of $\varepsilon^0$. Note that in (4.2) there are no stress components which decay rapidly will distance from the edges $y = 0$ and $y = a/b$, corresponding to the solution $\tau^g_{yz0}$. The factor $\varepsilon^\alpha$ is written before the last terms in each equation where $\alpha$ is a certain arbitrary exponent introduced due to the fact that $\tau^g_{yz0}$ is the general solution.

We will assume in Eqs (4.2) that $\alpha = -1$. Then, retaining only the principal terms, we obtain conditions of the form

$$\sigma_x(w_0) = 0, \quad \tau_{xy}(w_0) = 0, \quad \tau_{xz}(w_0) + \tau^p_{xz0} + \tau^g_{xz0} = 0 \tag{4.3}$$

It is well known [5] that the solution of the bi-harmonic equation for a rectangular plate, two opposite edges of which are freely supported and the other two are free, contains four arbitrary constants. They are determined from the first two conditions of (4.3). The last condition of (4.3), in which the first two terms are known now, serves to determine the constant for the decaying component of the general solution (3.12).

## 5. CONCLUSION

We will compare the results obtained with those obtained in [6, 7]. The classical theory of plates is formulated in papers by Poisson, Kirchhoff, Thomson and Tait. It is assumed that the normal element of the plate remains normal to the median surface and non-deformable during the deformation process, i.e.

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} \ (u \leftrightarrow v, x \leftrightarrow y), \quad \frac{\partial w}{\partial z} = 0$$

In the terms of the present paper these hypotheses can be written as

$$\tau_{xz0} = 0 \ (x \leftrightarrow y), \quad w_0 = w_0(x, y)$$

i.e. *a priori* hypotheses of the theory of plate [6] are *a priori* or initial approximations of the method of simple iterations in this paper. Furthermore, they are part of the initial approximation (3.3), which leads to the construction of the $w$-process. Other theories, called the Timoshenko, Reissner or Reissner–Mindlin type theories can be obtained by introducing one hypothesis, namely, the hypothesis of the invariance of the thickness of the plate during deformation. In the terms used here this is written as

$$w = w_0(x, y), \quad \tau_{xz} = \tau_{xz0}(x, y, z) \ (x \leftrightarrow y) \tag{5.1}$$

The difference between the choice of the quantities of the initial approximation (5.1) and (3.3) consists in the equations of the boundary layer, as was shown in the example of a strip. For instance, the equation of the boundary layer in partial derivatives with respect to $x$ and $z$ will correspond to the choice of (5.1) at the edge $y$ = const whereas Eq. (3.1) in which derivatives with respect to $y$ should be neglected, will correspond to the choice of (3.3). The asymptotic way is the straightest way of obtaining the solution using the $w$ and $\tau$-processes. The asymptotic process is easily formulated for establishing relations of the type (2.1). They can be determined by reviewing the sequence of the calculations by the method of simple iterations. The asymptotic process gives an equation of the boundary layer in partial derivatives with respect to $x$ and $z$, as shown in Section 2 using the example of a strip. The same situation also occurs for the initial approximations (3.3) and (5.1). In the case of (5.1) the solution of the boundary-layer equations can be represented in the form of an expansion in a Fourier series with respect to the thickness of the plate. Confining ourselves to the first term of the series we obtain a solution which is practically identical with the solution corresponding to the initial approximation (3.3).

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